GOSTS

Defining the Ground Model

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Introduction

The definability of the ground model is simultaneously very intuitive and somewhat unintuitive.

- For any given $\mathbb{P}$-name τ, we can define what it means to have $p \Vdash$ "$\tau \in \check{V}$":

$$\{q \leqslant p : \exists x\ (q \Vdash \text{“}\tau = \check{x}\text{”})\} \text{ is dense below } p.$$

- So it kind of makes sense that we could collect all this information together to define V in $V[G]$:

$$V = \{x : \exists \tau \in V^{\mathbb{P}}\ \exists p \in G\ (x = \tau_G \wedge p \Vdash \text{“}\tau \in \check{V}\text{”})\}.$$

- The issue here is that $V[G]$ might not be able to separate $\mathbb{P}$-names in $V[G]$ from $\mathbb{P}$-names in V if it doesn't know what V is.

Introduction

To motivate why the definability of the ground model is unintuitive, consider Cohen forcing.

- Iterated forcing lets us go two steps in one: $V[G] = V[H][K]$ where K is generic over $V[H]$.
- So is V the ground model of $V[G]$ or is $V[H]$ the ground model?
- Even if we say what the poset we're extending by, it's still not obvious: $\mathrm{Add}(\omega, 1) \cong \mathrm{Add}(\omega, 1) \times \mathrm{Add}(\omega, 1)$.
- Any Cohen generic extension $V[G] = V[H][K]$ where K is Cohen generic over $V[H]$.
- So even if $V[G]$ knows it's an extension by $\mathbb{P}$, it's still not obvious what the ground model should be.

Introduction

So the actual statement of the theorem requires more than just knowing the poset.

Theorem

Let G be $\mathbb{P}$-generic over $V \models \mathsf{ZFC}$. Therefore V is first-order definable in $V[G]$ from the parameter $\wp(|\mathbb{P}|)^V$.

- If we consider cohen forcing, $\mathbb{P} = \mathrm{Add}(\omega, 1)$, $|\mathbb{P}| = \aleph_0$.
- If we factor $V[G] = V[H][K]$, we have that $\wp(\aleph_0)^V \neq \wp(\aleph_0)^{V[H]}$.
- Thus $V[G]$ as a forcing extension of both can still determine its ground model so long as you say what the reals you already have are.
- Note that AC (and powerset) are essential here. There are some results about weakening these hypotheses, but they cannot be removed entirely.

Introduction

Part of the reason why AC is necessary is the following fact.

Lemma

Let $V, V' \vDash \mathsf{ZFC}$ be two transitive models. Suppose

$$\{x \in V : x \subseteq \mathrm{Ord}\} = \{x \in V' : x \subseteq \mathrm{Ord}\}.$$

Therefore $V = V'$.

- If we merely have $V, V' \vDash \mathsf{ZF}$, then this result does not hold.
- This result is necessary for establishing the uniqueness of our definition of the ground model.
- The proof of this isn't too difficult, and highlights the technique of coding sets into sets of ordinals.

Introduction

Lemma

Let $V, V' \models$ ZFC be two transitive models. Suppose

$$\{x \in V : x \subseteq \mathrm{Ord}\} = \{x \in V' : x \subseteq \mathrm{Ord}\}.$$

Therefore $V = V'$.

Proof.

- Note that $x = y$ iff $\mathrm{trcl}(\{x\}) = \mathrm{trcl}(\{y\})$. Suppose $x \in V$.
- $\langle \mathrm{trcl}(\{x\}), \in \rangle$ is a well-founded structure and by AC, has size $\kappa \in \mathrm{Ord}$.
- Hence $\langle \mathrm{trcl}(\{x\}), \in \rangle \cong \langle \kappa, R \rangle$ for some $R \subseteq \kappa$ by coding pairs of ordinals into single ordinals.
- So $R \in V \cap V'$ and V' is able to decode R into a transitive set by a Mostowski collapse.
- Uniqueness of the collapse ensures the set is $\mathrm{trcl}(\{x\})$, so $x \in \mathrm{trcl}(\{x\}) \in V'$ as desired. $\dashv$

Overview

We will make use of a similar uniqueness property.

Theorem

Let $W \vDash \mathsf{ZFC}$ be transitive with regular cardinal $\delta \in W$. Therefore, up to agreement on $\mathcal{P}(\delta)$, there is a unique inner model $V \subseteq W$ such that

- *V, W have the δ-covering and δ-approximation properties; and*
- $(\delta^+)^V = (\delta^+)^W$.

- More precisely, if V, V' both satisfy the above with $\mathcal{P}(\delta)^V = \mathcal{P}(\delta)^{V'}$ then $V = V'$.
- By factoring $\mathbb{P}$ appropriately, we get that $V, V[G]$ have the $|\mathbb{P}|^+ = \delta$-covering and δ-approximation properties with the same δ^+.
- Hence we (more or less) define V as the unique such model with $\mathcal{P}(\delta)$ equal to $\mathcal{P}(\delta)^V$.

The Uniqueness Property

Definition

Let $V \subseteq W$ be transitive models of (fragments of) ZFC. Let δ be a cardinal of W.

- V, W have the δ-*covering property* iff for all $A \in W$, if $A \subseteq V$ with $|A|^W < \delta$, then $A \subseteq A'$ for some $A' \in V$ with $|A'|^V < \delta$.
- V, W have the δ-*approximation property* iff for all $A \in W$, if $A \subseteq V$ and

$$\forall x \in V\ (|x|^V < \delta \rightarrow A \cap x \in V),$$

then $A \in V$.

Basically,

- (δ-covering) we can *cover* $< \delta$-sized sets of W with $< \delta$-sized sets in V, and
- (δ-approximation) If every $< \delta$-sized subset of A is in V then $A \in V$ (even if A is very large).

The Uniqueness Property

Definition

Let $V \subseteq W$ be transitive models of (fragments of) ZFC. Let δ be a cardinal of W.

- (δ-covering) we can *cover* $< \delta$-sized sets of W with $< \delta$-sized sets in V, and
- (δ-approximation) If every $< \delta$-sized subset of $A \in W$ is in V then $A \in V$ (even if A is very large).

So let's prove the uniqueness property.

Theorem

Let $W \models$ ZFC be transitive with regular cardinal $\delta \in W$. Therefore, up to agreement on $\mathcal{P}(\delta)$, there is a unique inner model $V \subseteq W$ such

- *V, W have the δ-covering and δ-approximation properties; and*
- $(\delta^+)^V = (\delta^+)^W$.

The Uniqueness Property

So let $V, V' \subseteq W$ be transitive models such that

- δ is regular;
- $\mathcal{P}(\delta)^V = \mathcal{P}(\delta)^{V'}$;
- V, W and V', W have the δ-covering and δ-approximation properties; and
- All three calculate δ^+ in the same way.

So having size δ and $\leq \delta$ is the same in each model.
It suffices to show V and V' have the same sets of ordinals. First, we show the following.

Claim

If $A \subseteq \mathrm{Ord}$ is in W with $|A|^W < \delta$, then there's a $B \in V \cap V'$ of size $\leq \delta$ such that $A \subseteq B$.

The Uniqueness Property

Claim

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Proof.

- Write $A \subseteq \alpha$. We get some $B_0 \in V$ covering A of size $< \delta$, still with $B_0 \subseteq \alpha$.
- Then we get a $B_1 \in V'$ coverig $B_0 \in V \subseteq W$ of size $< \delta$.
- Continue to get an increasing sequence $\langle B_n : n < \omega \rangle$ with $B_n \in V$ for even n and $B_n \in V'$ for odd n.
- The union B_ω might not be in $V \cap V'$ (the sequence is formed in W).
- But this is fine: the regularity of δ implies $|B_\omega| < \delta$ in W and so we can continue to get $B_{\omega+1} \in V$, etc.
- This gives a sequence $\langle B_\xi \subseteq \alpha : \xi < \delta \rangle$ with cofinally many in V, cofinally many in V'.

The Uniqueness Property

Claim

If $A \subseteq \mathrm{Ord}$ is in W with $|A|^W < \delta$, then there's a $B \in V \cap V'$ of size $\leq \delta$ such that $A \subseteq B$.

Proof.

- So take $B_\delta = \bigcup_{\xi<\delta} B_\xi$ which extends the original A.
- Note $|B_\delta| \leq \delta$. So we want $B_\delta \in V \cap V'$.
- (δ-approximation) If every $< \delta$-sized $x \in V$ has $x \cap B_\delta \in V$ then $B_\delta \in V$.
- Any approximation $x \cap B_\delta$ of size $< \delta$ with $x \in V$ is actually just $x \cap B_\xi$ for sufficiently large $\xi < \delta$ (each element of x appears by some stage $< \delta$ and there are $< \delta$-many elements).
- δ-approximation for V, W gives that $B_\delta \in V$, and the same idea for V' holds. $\dashv$

The Uniqueness Property

Claim

If $A \subseteq \mathrm{Ord}$ is in W with $|A|^W < \delta$, then there's a $B \in V \cap V'$ of size $\leq \delta$ such that $A \subseteq B$.

We want V, V' to have the same sets of ordinals. Suppose $A \subseteq \mathrm{Ord}$ with $A \in V$.

- Suppose $|A|^V < \delta$. We want to code A as a subset of δ since $\mathcal{P}(\delta)^V = \mathcal{P}(\delta)^{V'}$.
- By the claim, there's a $B \in V \cap V'$ of size $\leq \delta$ covering A.
- The increasing enumeration of B has length $< \delta^+$.
- We can code this length $\beta < \delta^+$ as a well-order of δ.
- This well-order as a subset of $\delta \times \delta$ can be coded by a subset $C \subseteq \delta$.
- So we can enumerate $B = \{b_\alpha : \alpha \in C\}$ in both V and V'.
- A is then coded by $\{\alpha \in C : b_\alpha \in A\} \in \mathcal{P}(\delta)^{V'}$.
- This subset is in V' and V' can decode all of this with $B \in V'$ to get $A \in V'$.

The Uniqueness Property

Claim

If $A \subseteq \mathrm{Ord}$ is in W with $|A|^W < \delta$, then there's a $B \in V \cap V'$ of size $\leq \delta$ such that $A \subseteq B$.

We want V, V' to have the same sets of ordinals. Suppose $A \subseteq \mathrm{Ord}$ with $A \in V$.

- So now suppose $|A|^V \geq \delta$.
- If $x \in V'$ has size $< \delta$, so too does $x \cap \mathrm{Ord}$.
- $< \delta$-sized sets of ordinals are shared so $x \in V$ and thus $A \cap x \in V$.
- Again, $A \cap x \in V$ has size $< \delta$ so that $A \cap x \in V'$.
- δ-approximation applied to V', W then gives $A \in V'$.

So V, V' have the same sets of ordinals. By the lemma, $V = V'$. $\dashv$

The Uniqueness Property

Theorem

Let $W \models \mathsf{ZFC}$ be transitive with regular cardinal $\delta \in W$. Therefore, up to agreement on $\wp(\delta)$, there is a unique inner model $V \subseteq W$ such

- *V, W have the δ-covering and δ-approximation properties; and*
- $(\delta^+)^V = (\delta^+)^W$.

- The question now becomes what examples $V \subseteq W$ do we have with both the δ-covering and δ-approximation properties?
- The idea is to consider preorders of the form $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ is non-trivial of size $< \delta$ and $\dot{\mathbb{Q}}$ is $< \delta$-strategically closed.
- This is a weakening of the notion of admitting a *gap* which requires $\dot{\mathbb{Q}}$ to be $\leq \delta$-strategically closed.

δ-strategically closed preorders

Strategic closure has been mentioned before, but let's review the definition.

Definition

Let $\alpha \in \mathrm{Ord}$ and $\mathbb{P}$ a preorder. The game $G^{\alpha}_{\mathbb{P}}$ is the game

$$\begin{array}{lcccccccccc} \mathbf{I}: & p_0 = \mathbb{1}^{\mathbb{P}} & & p_2 \in \mathbb{P} & & \cdots & & p_\omega & & \cdots \\ \mathbf{II}: & & p_1 \in \mathbb{P} & & p_3 & & \cdots & & p_{\omega+1} & \end{array}$$

of length α where **I** wins iff the result is a descending chain of conditions.

- $\mathbb{P}$ is α-strategically closed iff **I** wins $G^{\alpha}_{\mathbb{P}}$.
- $\mathbb{P}$ is $\leq \alpha$-strategically closed iff **I** wins $G^{\alpha+1}_{\mathbb{P}}$.
- $\mathbb{P}$ is $< \alpha$-strategically closed iff $\mathbb{P}$ is $\leq \beta$-strategically closed for each $\beta < \alpha$.

- So clearly if $\mathbb{P}$ is $< \kappa$-closed, $\mathbb{P}$ is $< \kappa$-strategically closed.
- Usually we play the role of **II**, having **I** pick conditions at limit stages.

δ-strategically closed preorders

Many of the arguments about closed posets generalize to strategically closed ones, but with a little more effort.

Result

For any infinite cardinal δ, if $\mathbb{P}$ is $\leq \delta$-strategically closed, then $\mathbb{P}$ is $\leq \delta$-distributive.

Proof.

- Let $\mathcal{D}$ be a collection of $\leq \delta$-many open, dense sets. We want to show $\bigcap \mathcal{D}$ is dense.
- Enumerate $\mathcal{D} = \{D_\alpha : \alpha < \delta\}$. Normally, we'd use closure to pick an extension of an arbitrary $p \in \mathbb{P}$ in each dense set.
- With access to only *half* of the extensions, we do this slowly:
- Let τ be a strategy for $\mathbf{I}$ in $G^\delta_\mathbb{P}$. Consider the play where $\mathbf{I}$ uses τ:

$$\begin{array}{lccccccc} \mathbf{I}: & p_0 = \mathbb{1} & & p_2 & & \cdots & p_\delta \\ \mathbf{II}: & & p_1 \in D_0 & & p_3 \in D_1 & & \end{array}$$

- By openness, $p_\delta \in D_\alpha$ for each $\alpha < \delta$, as desired. $\dashv$

δ-strategically closed preorders

Result

For any infinite cardinal δ, if $\mathbb{P}$ is $\leq \delta$-strategically closed, then $\mathbb{P}$ is $\leq \delta$-distributive.

- A similar idea tells us that being $< \delta$-strategically closed implies $\mathbb{P}$ is $< \delta$-distributive.
- In general, a good way to show properties of strategically closed preorders is to show the result for *closed* ones, and generalize.
- Strategic closure is mostly brought up for the following theorem. It's more than we need, but it's useful for many reasons.

Lemma

Suppose δ is a cardinal;

- *$\mathbb{P}$ is a non-trivial preorder of size $< \delta^+$;*
- *$\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $< \delta^+$-strategically closed"; and*
- *G is $\mathbb{P}$-generic over V.*

Therefore $V, V[G]$ have the δ^+-covering and δ^+-approximation properties.

The Covering and Approximation Properties

Lemma

Suppose δ is a cardinal;

- $\mathbb{P}$ *is a non-trivial preorder of size* $\leq \delta$*;*
- $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ *is* $\leq \delta$*-strategically closed"; and*
- G *is* $\mathbb{P} * \dot{\mathbb{Q}}$*-generic over* V*.*

Therefore $V, V[G]$ *have the* δ^+*-covering and* δ^+*-approximation properties.*

- Any preorder factors in this way by taking $\delta = |\mathbb{P}|$ and $\mathbb{Q}$ as trivial.
- We mostly care about the δ^+-approximation property because it's harder to show.
- The proof of this is actually a bit involved and often uses a kind of tree of conditions that Hamkins originally used.
- I don't particularly like it, so I'll follow Mitchell's proof.

Exercise

Show the δ^+*-covering property holds because it holds for each step of the iteration.*

The Covering and Approximation Properties

- (δ^+-approximation) If every $\leq \delta$-sized $x \in V$ has $x \cap A \in V$ then $A \in V$.
- Let $A \in V[G]$ with $A \subseteq V$. Without loss of generality, take $A \subseteq \mathrm{Ord}$ just by enumerating sets in V and approximating the corresponding indices.
- Let $A \subseteq \kappa$ for some cardinal κ. Suppose $A \cap X \in V$ for any δ-sized subset $X \subseteq \kappa$ in V.
- Let $\dot{A}$ be a name for A.
- Consider an (uncollapsed) skolem hull $M = \mathrm{Hull}^{\mathrm{H}_\theta^V}(\mathbb{P} \cup \{\kappa, \mathbb{P} * \dot{\mathbb{Q}}, \dot{A}\})$ of size $\leq \delta$ so that $\mathbb{P} \subseteq M \preccurlyeq \mathrm{H}_\theta$ for some sufficiently large θ.
- Note $M \cap \kappa \in V$ has size $\leq \delta$ so that $A \cap M \in V$.
- We will show there's a condition $\mathbb{P} * \dot{\mathbb{Q}}$ that decides "$\check{\alpha} \in \dot{A}$" for every $\alpha \in M \cap \kappa$.
- In fact, the condition will decide every element of $\dot{A}$ and thus defines $A \in V$.

The Covering and Approximation Properties

- $M = \mathrm{Hull}^{H_\theta^V}(\mathbb{P} \cup \{\kappa, \mathbb{P} * \dot{\mathbb{Q}}, \dot{A}\})$ of size $\leq \delta$
- To get a grip on deciding membership in $\dot{A}$ in M, note that for $\alpha < \kappa$, if $\langle p, \dot{q} \rangle$ doesn't decide "$\check{\alpha} \in \dot{A}$" then we can force either way by splitting p.
- Importantly, we *don't* need to split $\dot{q}$.

Claim

If $\langle p, \dot{q} \rangle$ doesn't decide "$\check{\alpha} \in \dot{A}$", then there are extensions $p_0^*, p_1^* \leq p$ and $\mathbb{1}^{\mathbb{P}} \Vdash$ "$\dot{q}^* \leq \dot{q}$" such that

$$\langle p_0^*, \dot{q}^* \rangle \Vdash \text{"}\check{\alpha} \in \dot{A}\text{"} \quad \textit{and} \quad \langle p_1^*, \dot{q}^* \rangle \Vdash \text{"}\check{\alpha} \notin \dot{A}\text{"}.$$

Proof.

Basically, when extending to decide, just find a canonical name extending the $\dot{q}$s which is interpreted differently by extensions of p. $\dashv$

The Covering and Approximation Properties

Claim

If $\langle p, \dot{q}\rangle$ doesn't decide "$\check{\alpha} \in \dot{A}$", then there are extensions $p_0^*, p_1^* \leqslant p$ and $\mathbb{1}^{\mathbb{P}} \Vdash$ "$\dot{q}^* \leqslant \dot{q}$" such that

$$\langle p_0^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \in \dot{A}\text{”} \quad \text{and} \quad \langle p_1^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \notin \dot{A}\text{”}.$$

- So we can continually use the claim to extend $\dot{q} \in M$.
- Then use strategic closure to get a single $\dot{q}$ that works: we don't need to extend $\dot{q}$ at all as above!

Claim

There is a $\dot{q} \in M$ where for every $p \in \mathbb{P}$, there are extensions $p_0^*, p_1^* \leqslant p$ and an ordinal $\alpha \in M$ such that

$$\langle p_0^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \in \dot{A}\text{”} \quad \text{and} \quad \langle p_1^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \notin \dot{A}\text{”}.$$

The Covering and Approximation Properties

Claim

There are extensions $p_0^*, p_1^* \leqslant p$ and $\mathbb{1}^{\mathbb{P}} \Vdash$ "$\dot{q}^* \leqslant \dot{q}$" such that

$$\langle p_0^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \in \dot{A}\text{”} \quad \text{and} \quad \langle p_1^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \notin \dot{A}\text{”}.$$

Claim

There is a $\dot{q} \in M$ where for every $p \in \mathbb{P}$, there are extensions $p_0^*, p_1^* \leqslant p$ and an ordinal $\alpha \in M$ such that

$$\langle p_0^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \in \dot{A}\text{”} \quad \text{and} \quad \langle p_1^*, \dot{q}^*\rangle \Vdash \text{“}\check{\alpha} \notin \dot{A}\text{”}.$$

Proof.

- If $\langle p, \dot{q}\rangle$ doesn't work, extend $\dot{q}$ to $\dot{q}_1$ as per the first claim.
- Extending in this way preserves the previous decisions of its extensions.
- Then we can consider the next element of $\mathbb{P}$ paired with $\dot{q}_1$ and extend again if we haven't finished.
- Since $|\mathbb{P}| < \delta^+$ in M, the $< \delta^+$-strategic closure of $\dot{\mathbb{Q}}$ gives a name $\dot{q}$ that works. ⊣

The Covering and Approximation Properties

Claim

There is a $\dot{q} \in M$ where for every $p \in \mathbb{P}$, there are extensions $p_0^*, p_1^* \leqslant p$ and an ordinal $\alpha \in M$ such that

$$\langle p_0^*, \dot{q}^* \rangle \Vdash \text{“}\check{\alpha} \in \dot{A}\text{”} \quad \text{and} \quad \langle p_1^*, \dot{q}^* \rangle \Vdash \text{“}\check{\alpha} \notin \dot{A}\text{”}.$$

- Really all this means is that $\dot{q} \in M$ can be used to determine $A \cap M$.
- Let $a = A \cap M \in V$ and extend $\langle \mathbb{1}^{\mathbb{P}}, \dot{q} \rangle \geqslant \langle p^*, \dot{q}^* \rangle \Vdash$ “$\dot{A} \cap \check{M} = \check{a}$”.
- It follows that $\langle p^*, \dot{q}^* \rangle$ decides every element of $A \cap M$.
- To see it decides all of A, if it doesn't, then neither does $\langle p^*, \dot{q} \rangle$.
- But by the claim, there are then extensions $p_0^*, p_1^* \leqslant p^*$ and an ordinal $\alpha \in M$ such that
$$\langle p_0^*, \dot{q} \rangle \Vdash \text{“}\check{\alpha} \in \dot{A}\text{”} \quad \text{and} \quad \langle p_1^*, \dot{q} \rangle \Vdash \text{“}\check{\alpha} \notin \dot{A}\text{”}.$$
- Hence $\langle p_0^*, \dot{q}^* \rangle$ and $\langle p_1^*, \dot{q}^* \rangle$ decide in the same way, both extending $\langle p^*, \dot{q}^* \rangle$.
- But $\langle p^*, \dot{q}^* \rangle$ already decided “$\check{\alpha} \in \dot{A}$” since $\alpha \in M$, a contradiction.
- Hence $\langle p^*, \dot{q}^* \rangle$ decides all of A in V.

Defining the Ground Model

Great, so how do we put it all together?

Theorem (Main Theorem)

Let G be $\mathbb{P}$-generic over $V \vDash \mathsf{ZFC}$. Therefore V is first-order definable in $V[G]$ from the parameter $\wp(|\mathbb{P}|)^V$.

Proof.

- Let $\dot{\mathbb{Q}}$ be trivial and $\delta = |\mathbb{P}|$.
- By the previous theorem, V, $V[G]$ have the δ^+-covering and δ^+-approximation properties.
- The δ^+-covering property ensures δ^{++} is calculated properly.

Defining the Ground Model

Theorem (Uniqueness theorem)

Let $W \models \mathsf{ZFC}$ *be transitive with regular cardinal* $\delta \in W$*. Therefore, up to agreement on* $\mathcal{P}(\delta)$*, there is a unique inner model* $V \subseteq W$ *such*

- V, W *have the* δ*-covering and* δ*-approximation properties; and*
- $(\delta^+)^V = (\delta^+)^W$.

Proof of the Main Theorem.

- Using reflection, there's actually a club of ordinals λ such that
  1. $\mathrm{V}_\lambda^V, \mathrm{V}_\lambda^{V[G]}$ have the δ^+-covering and δ^+-approximation properties;
  2. $\mathrm{V}_\lambda^V, \mathrm{V}_\lambda^{V[G]}$ satisfy enough set theory for the uniqueness theorem to go through.
- As a result, we can define V_λ^V from $\mathcal{P}(\delta^+)^V$ in $V[G]$ as the unique transitive model M such that
  1. $M \models \mathsf{ZFC}^*$ enough set theory;
  2. $\lambda = M \cap \mathrm{Ord}$;
  3. $\mathcal{P}(\delta^+)^M = \mathcal{P}(\delta^+)^V$;
  4. $M, \mathrm{V}_\lambda^{V[G]}$ have the δ^+-covering and δ^+-approximation properties.

Defining the Ground Model

Proof of the Main Theorem.

- We can define V_λ^V from $\mathcal{P}(\delta^+)^V$ in $V[G]$ as the unique transitive model M such that
  1. $M \models \mathsf{ZFC}^*$ enough set theory;
  2. $\lambda = M \cap \mathrm{Ord}$;
  3. $\mathcal{P}(\delta^+)^M = \mathcal{P}(\delta^+)^V$;
  4. M, $V_\lambda^{V[G]}$ have the δ^+-covering and δ^+-approximation properties.
- We can then define V as the sets which appear in such M.
- Note that this uses $\mathcal{P}(\delta^+)$ as the parameter instead of $\mathcal{P}(\delta)$.
- This isn't an issue by the δ^+-approximation property.
- Any A a δ-sized subset of δ^+ is bounded by some $\alpha < \delta^+$.
- Again, we can code A as a subset of δ and having enough set theory to decode such an enumeration determines A.
- So $\mathcal{P}(\delta)^V$ suffices to determine $\mathcal{P}(\delta^+)$. $\dashv$

Defining the Ground Model

Theorem (Main Theorem)

Let G be $\mathbb{P}$-generic over $V \vDash \mathsf{ZFC}$. Therefore V is first-order definable in $V[G]$ from the parameter $\wp(|\mathbb{P}|)^V$.

There are a lot of nice consequences of this, but mostly it allows us to make many definitions that relate the ground model and generic extension precise. For example, we have that for any $\mathbb{P}$ of size $< \kappa$, a measurable cardinal,

$$\mathbb{1}^{\mathbb{P}} \Vdash \text{“Every measure extends and lifts from a measure in } \check{V}\text{”}.$$

Such ideas are important in the preservation of large cardinal properties. (And are discussed in places like Cummings’ chapter in the Handbook.)

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